

Loop cosmological implications of a non-minimally coupled scalar field.

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Non-minimal actions with matter represented by a scalar field coupled to gravity are considered in the context of a homogeneous and isotropic universe. The coupling is of the form $-\frac{1}{2}\xi\phi^2R$. The possibility of successful inflation is investigated taking into account features of loop cosmology. For that end a conformal transformation is performed. That brings the theory into the standard minimally coupled form (Einstein frame) with some effective field and its potential. Both analytical and numerical estimates show that a negative coupling constant is preferable for successful inflation. Moreover, provided fixed initial conditions, larger $|\xi|$ leads to a greater number of e -folds. The latter is obtained for a reasonable range of initial conditions and the coupling parameter and indicates a possibility for successful inflation.

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I. INTRODUCTION

Modern astronomical observations gave rise to a number of important conceptual issues, which can be successfully resolved by the mechanism called *inflation*. According to the inflationary paradigm [1, 2], there has to have been an epoch, when the Universe was undergoing an accelerated expansion; such that the size of the Universe increased by a factor of about e^{60} (60 e -foldings) to agree with observations.

This scenario can be realized within the framework of the Friedmann-Robertson-Walker model, if the matter is represented by a self-interacting scalar field, the *inflaton*, which is minimally coupled to gravity. The mechanism considers so called *slow-roll approximation*, when the inflaton rolls down its potential hill towards the potential minimum, but, at the same time, remains far away from the minimum. It is in the vicinity of the minimum of the potential, where inflation stops. For the simplest choice of a quadratic field potential with an arbitrary mass of the inflaton, one would obtain 60 e -foldings if the slow-roll regime starts off at $\phi \approx 3M_P$ and ends when the field is essentially zero. The question as to how the inflaton gets to that value before the slow-roll phase is left for a quantum theory to answer.

Without a quantum theory taking into account gravity at hand, one could heuristically argue along the lines of Linde's chaotic inflationary scenario [3]. The idea was that vacuum fluctuations of a scalar field would be distributed randomly up to Planckian energy densities. Regions with sufficient values of ϕ would inflate resulting in a flat FRW-universe. We shall see that, however insightful, this paradigm imposes severe restrictions on the likelihood of inflation in non-minimally coupled models, with a coupling of the form $-\frac{1}{2}\xi\phi^2R$ [4, 5]. Specifically, the

coupling constant needs to be very small for successful inflation: $\xi < 10^{-12}$ for positive coupling and $|\xi| < 10^{-3}$ for negative coupling.

Loop Quantum Cosmology (LQC) [6] has over the last years resolved some long-standing issues of cosmology and general relativity in whole and suggested an alternative mechanism to chaotic inflation. As has been shown in [7, 8], LQC leads to effective quantum modifications of the classical Friedmann equations, such that gravity becomes repulsive at small scales and introduces an 'anti-friction' term into the Klein-Gordon equation for the scalar, thus driving the inflaton up its potential hill [9, 10]. Therefore even small vacuum fluctuations of ϕ will be amplified during the quantum-corrected (effective) phase of evolution. In the light of this, much smaller field initial values are required to start a successful classical inflation. This fact significantly relaxes the restrictions on the coupling constant of a non-minimally coupled theory. Moreover, it is possible to study the inflaton's 'climbing-up' systematically, provided some initial fluctuations occur in the field and its canonically conjugate momentum.

The paper is organized as follows. We first review the conventional model for a non-minimally coupled scalar field. Then, following [4, 5], we perform a conformal transformation, which allows us to recast the action in the minimally coupled form in terms of a redefined field variable with some effective potential. In section three, we investigate the slow-roll regime and derive the number of e -folds as a function of the value of the inflaton right before inflation. Then, in section four, we turn to investigating the inflaton's 'climbing', that is how small initial fluctuations get amplified during the quantum-corrected phase and the following classical evolution. The estimates for the maximum field value are mostly obtained analytically, up to a certain point when one is forced to solve a transcendental equation. The results are then compared with the numerical solutions of the Hamilton equations of motion in section five. The comparison indicates that the analytical method provides a decent preci-

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sion and hence can be efficiently used for the analysis of the range of initial conditions and parameters that lead to a sufficiently long inflationary phase.

II. ACTION FOR A NON-MINIMALLY COUPLED SCALAR AND CONFORMAL TRANSFORMATION

The simplest form of a non-minimally coupled action is given by

$$S[g_{ab}, \phi] = \int d^4x \sqrt{-g} \left(\frac{f(\phi)}{2\kappa} R - \frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi - U(\phi) \right) \quad (1)$$

where $g \equiv \det(g_{ab})$, $\kappa = 8\pi G$, R is the scalar curvature of the metric and $U(\phi)$ is the self-interacting potential of the matter scalar field ϕ . The curvature term is coupled to matter through the coupling function $f(\phi)$ and is manifestly Lorentz invariant. There are some obvious restrictions on the coupling function. First of all, as we shall see, it enters the symplectic structure for the gravitational (connection) variables. Thus it must not vanish anywhere. Moreover, one can view this function as defining an effective gravitational constant κ , in the sense

$$\kappa_{\text{eff}} = \kappa / f(\phi) \quad (2)$$

Hence we require $f(\phi)$ to be always positive. Note also that the scalar field is expected to change sign, e.g. during the reheating epoch. It is then natural to restrict $f(\phi)$ to be an even function of ϕ . Finally, the coupling function must satisfy the limit $f(\phi) \rightarrow 1$ when $\phi \rightarrow 0$, so that the standard normalization is recovered in the absence of a scalar field.

Given the restrictions above, the simplest form of the coupling function is

$$f(\phi) = 1 - \sigma \phi^2. \quad (3)$$

(One can think of this expression as the Taylor expansion up to the quadratic order. We are interested in the case of a weak coupling, so the latter should be a good approximation. Note also that coupling of higher degree would require a dimensionful coupling constant, while ξ is unitless.) In fact, this is the form of coupling function that is normally considered in the literature (see, for example, [11]). We will refer to σ as the *coupling strength* (not to be confused with the *coupling constant* $\xi \equiv \frac{\sigma}{\kappa}$ of [4]). Note that zero σ corresponds to the case of minimal coupling. If $\sigma > 0$, the coupling function can, in principle, become zero and even negative if the value of the scalar field exceeds the critical value

$$\phi_{\text{crit}} := \frac{1}{\sqrt{|\sigma|}} \quad (4)$$

We have deliberately written the absolute value of the coupling strength because, as we shall see later on, it is also of importance in the case of negative coupling.

In this paper, we restrict our attention to a flat homogeneous and isotropic model, described by the FRW-metric

$$ds^2 = -dt^2 + a(t)^2 d\vec{r}^2 \quad (5)$$

Using a conformal transformation

$$\tilde{g}_{ab} := f(\phi) g_{ab} \quad (6)$$

the action (1) can be recast in the form [4]

$$S[\tilde{g}_{ab}, \phi] = \int dt \left[\frac{3}{\kappa} (a^2 \ddot{a} + a \dot{a}^2) + a^3 \left(\frac{1}{2} F(\phi)^2 \dot{\phi}^2 - \tilde{V}(\phi) \right) \right] \quad (7)$$

where we have introduced the effective potential

$$\tilde{V}(\phi) := \frac{U(\phi)}{f(\phi)^2} \quad (8)$$

and

$$F(\phi)^2 := \frac{1 - \sigma \phi^2 (1 - \frac{6\sigma}{\kappa})}{f(\phi)^2} \approx \frac{1}{f(\phi)}. \quad (9)$$

The last approximation holds as long as we are interested in weak coupling such that $\sigma \sim (10^{-3} \div 10^{-2})\kappa$; then $(1 - \frac{6\sigma}{\kappa})$ can be set to 1 in this order of magnitude. Note that in the Hamiltonian formulation, there is a similar canonical transformation under which $F(\phi)^2 := \frac{1}{f(\phi)}$ exactly [12]. Consequently, the relation between the scalar fields (11), derived below, also becomes exact, which considerably simplifies the analysis even if $\frac{6\sigma}{\kappa} \neq 1$. The last step, that will bring the kinetic term into its canonical form (Einstein frame), is to redefine the field

$$\begin{aligned} \tilde{\phi} &:= \int d\phi F(\phi) \approx \int \frac{d\phi}{\sqrt{1 - \sigma \phi^2}} \\ &= \frac{1}{\sqrt{\sigma}} \begin{cases} \sin^{-1}(\sqrt{\sigma \phi}), & \text{if } \sigma > 0 \\ \sinh^{-1}(\sqrt{|\sigma| \phi}), & \text{if } \sigma < 0 \end{cases} \end{aligned} \quad (10)$$

Note that for a positive σ the quantity $\sqrt{\sigma} \phi$ must be less than unity, for $f(\phi)$ must not vanish and, in fact, be positive. From (10) we get

$$\sqrt{|\sigma|} \phi \approx \begin{cases} \sin(\sqrt{\sigma} \tilde{\phi}), & \sigma > 0 \\ \sinh(\sqrt{|\sigma|} \tilde{\phi}), & \sigma < 0 \end{cases} \quad (11)$$

Aiming at a loop quantization, we now have all the ingredients to proceed to the Hamiltonian formulation of the theory. For that end we introduce the phase space variables:

$$|\tilde{p}| := a^2, \quad \tilde{c} := \gamma \dot{a}, \quad \tilde{\phi} \quad \text{and} \quad \tilde{\pi} := p^{3/2} \dot{\tilde{\phi}} \quad (12)$$

here \tilde{p} and \tilde{c} are the gauge-invariant triad and connection components respectively, γ is the Barbero-Immirzi parameter [13, 14], and $\tilde{\pi}$ is the field momentum. Generally, the sign of \tilde{p} determines the orientation of the triad. From now on we assume that $\tilde{p} > 0$. Again, the

tilded variables stand for quantities written in the Einstein frame. For notational convenience, however, we will omit tildes until transforming back to the Jordan frame. In terms of newly defined variables the action (7) takes the form

$$S = \int dt \left[\frac{3}{\kappa\gamma} p\dot{c} + \pi\dot{\phi} - \mathcal{H} \right]. \quad (13)$$

with the Hamiltonian

$$\mathcal{H} = -\frac{3}{\kappa\gamma^2} \sqrt{p} c^2 + \frac{\pi^2}{2p^{3/2}} + p^{3/2} V(\phi) \quad (14)$$

From (13) we see that the pairs $\{p, c\}$ and $\{\phi, \pi\}$ are indeed canonically conjugate variables with the Poisson brackets

$$\{c, p\} = \frac{\kappa\gamma}{3}, \quad \{\phi, \pi\} = 1 \quad (15)$$

Before proceeding to inflation, let us comment on a phenomenological relation between the Einstein and Jordan frames. More specifically, eventually we are interested in the number of e-folds during the inflationary epoch

$$\tilde{N} := \ln \left(\frac{\tilde{a}_e}{\tilde{a}_b} \right) \quad (16)$$

where the subscripts stand for the initial (before inflation) and final (after inflation) values of the scale factor \tilde{a} . Suppose we have solved the equations of motion in the Einstein frame and obtained \tilde{a}_b and \tilde{a}_e . Then it is clear from (6) that the scale factors in the two frames are related by

$$\tilde{a} = \sqrt{f(\phi)} a \quad (17)$$

Therefore the Jordanian number of e-folds is

$$N \equiv \ln \left(\frac{a_e}{a_b} \right) = \ln \left(\frac{\sqrt{f(\phi_b)} \tilde{a}_e}{\sqrt{f(\phi_e)} \tilde{a}_b} \right) = \tilde{N} + \frac{1}{2} \ln \left(\frac{f(\phi_b)}{f(\phi_e)} \right) \quad (18)$$

At the end of inflation, the scalar field is essentially zero, while at the beginning of inflation it has its maximum value ϕ_{\max} . Thus, using (3), we can explicitly write

$$N = \tilde{N} + \frac{1}{2} \ln(1 - \sigma \phi_{\max}^2) \quad (19)$$

In principle, the beginning of inflation may occur when the scalar field is close to its critical value, such that $|\sigma \phi_{\max}| \approx 1$. In this case, a negative coupling $\sigma < 0$ would yield a negligible correction to the righthand side of (19). A positive coupling, on the contrary, might give a significant contribution resulting in a substantial difference between the Jordanian and Einsteinian number of e-folds.

III. INFLATION

In this section, we consider the inflationary mechanism called *slow-roll approximation* [2]. According to this model, the inflaton rolls down its potential hill towards the potential minimum, but, at the same time, remains far away from the minimum. In such a regime, a scalar field behaves as a cosmological constant. It is in the vicinity of the minimum of the potential, where inflation stops. We compute the number of e-folds as a function of the initial (maximum) value of the scalar field. We shall see that the result is generic irrespective of the sign of the coupling strength σ .

We start with investigating the equations of motion generated by the transformed Hamiltonian (14).

$$\begin{aligned} \dot{p} &= 2\sqrt{p} \frac{c}{\gamma} = \sqrt{\frac{2\kappa}{3}} \sqrt{\frac{\pi^2}{p} + 2p^2 V(\phi)} \\ \dot{\phi} &= \frac{\pi}{p^{3/2}} \\ \dot{\pi} &= -p^{3/2} V'(\phi) \end{aligned} \quad (20)$$

where the prime denotes a ϕ -derivative. In the first equation we have eliminated the connection c , using the Hamiltonian constraint (14) $H \approx 0$. We shall make an extensive use of these equations later on. For the moment, however, it is more convenient to consider the conventional Friedmann equations. The constraint equation, rewritten in terms of the scale factor and its time derivative, reads

$$\mathcal{H} = -\frac{3}{\kappa} a \dot{a}^2 + \frac{a^3}{2} \dot{\phi}^2 + a^3 V(\phi) \approx 0 \quad (21)$$

Dividing by a^3 , one gets the Friedmann equation:

$$H^2 = \frac{\kappa}{3} \left[\frac{\dot{\phi}^2}{2} + V(\phi) \right] \quad (22)$$

Here $H \equiv \dot{a}/a$ is the Hubble rate. Using the $\dot{\pi}$ -equation we derive the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \quad (23)$$

Finally, the Raychaudhuri equation can be obtained by taking the time derivative of (22) and substituting $\ddot{\phi}$ from (23):

$$\dot{H} = -\frac{\kappa}{2} \dot{\phi}^2 \quad (24)$$

From now on we stick with the slow-roll approximation that would ensure inflation. The potential domination (small kinetic terms)

$$\dot{\phi}^2 \ll V(\phi), \quad |\ddot{\phi}| \ll |3H\dot{\phi}| \quad (25)$$

is provided, if we require the slow-roll parameters to be small:

$$\begin{aligned}\epsilon &= \frac{1}{2\kappa} \left(\frac{V'}{V} \right)^2 \ll 1 \\ \eta &= \frac{1}{\kappa} \left(\frac{V''}{V} \right) \ll 1\end{aligned}\quad (26)$$

With the assumptions (25) the Friedmann and Klein-Gordon equations can be rewritten as

$$\begin{aligned}H^2 &\approx \frac{\kappa}{3} V(\phi) \\ 3H\dot{\phi} &\approx -V'(\phi)\end{aligned}\quad (27)$$

whereas the Raychaudhuri equation simply implies $\dot{H} \approx 0$, i.e. an exponential expansion of the scale factor. Dividing the first of the equations (27) by the second one, we can express the Hubble rate as

$$H \approx -\kappa \dot{\phi} \frac{V}{V'} \quad (28)$$

The number of e-folds is then given by

$$\begin{aligned}N &= \int_{t_b}^{t_e} H dt \approx -\kappa \int_{t_b}^{t_e} \frac{V}{V'} \dot{\phi} dt \\ &= -\kappa \int_{\phi_b}^{\phi_e} \frac{V}{V'} d\phi\end{aligned}\quad (29)$$

where ϕ_b and ϕ_e are the values of the inflaton before and after inflation respectively. The last expression tells us that as soon as the shape of the potential is specified, one can compute the number of e-folds. Therefore, it is now pertinent to analyze the form of the effective potential given by (8). The most common choice is either quadratic or quartic potential

$$U(\phi) = \frac{m^2}{2} \phi^2, \quad \text{or} \quad U(\phi) = \frac{\lambda}{4} \phi^4 \quad (30)$$

The corresponding effective potentials are

$$\begin{aligned}V(\phi) &= \frac{m^2}{2\sigma} \frac{\tan^2(\sqrt{\sigma}\phi)}{\cos^2(\sqrt{\sigma}\phi)}, \quad \text{and} \\ V(\phi) &= \frac{\lambda}{4\sigma^2} \tan^4(\sqrt{\sigma}\phi)\end{aligned}\quad (31)$$

if σ is positive, and

$$\begin{aligned}V(\phi) &= \frac{m^2}{2|\sigma|} \frac{\tanh^2(\sqrt{|\sigma|}\phi)}{\cosh^2(\sqrt{|\sigma|}\phi)}, \quad \text{and} \\ V(\phi) &= \frac{\lambda}{4\sigma^2} \tanh^4(\sqrt{|\sigma|}\phi)\end{aligned}\quad (32)$$

if σ is negative.

Fig. 1 shows the effective potentials for the quadratic original potential. If $\sigma > 0$ (Fig. 1), the effective potential is steeper than quadratic (punched line) and diverges

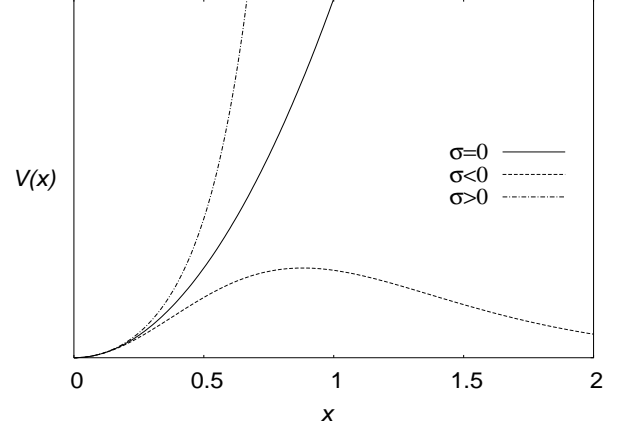


FIG. 1: The effective potential for $\sigma < 0$ (upper curve) and $\sigma > 0$ (lower curve). The solid curve represents the original quadratic potential. The argument of the potential is $x \equiv \sqrt{|\sigma|}\phi$ for $\sigma \neq 0$ and $x = \phi$ for $\sigma = 0$.

at $\sqrt{\sigma}\phi = \frac{\pi}{2}$ thus keeping the inflaton away from that point. Note that this is exactly the value of ϕ at which the coupling function goes to zero. On the contrary, the effective potential for a negative σ is bounded from above, goes below the quadratic potential (punched line) and has a maximum at $\sqrt{|\sigma|}\phi = \ln(1 + \sqrt{2})$. Remarkably, both these values of the tilded scalar correspond to the critical value of the original field (4). The existence of the maximum for a negative coupling constant implies that if the inflaton exceeds the critical value, it will roll down to the right, towards infinite values. The latter is unacceptable, as the inflaton is supposed to dissipate to zero after inflation. We therefore should restrict our attention to $\phi < \phi_{\text{crit}}$ for both positive and negative couplings.

Let us now retrieve the tildes over the conformally transformed variables, to distinguish them from the original ones, and calculate the number of e-folds

$$\begin{aligned}\tilde{N} &= -\kappa \int_{\tilde{\phi}_b}^{\tilde{\phi}_e} \frac{\tilde{V}}{\tilde{V}_{,\tilde{\phi}}} d\tilde{\phi} \\ &= -\kappa \int_{\phi_b}^{\phi_e} \frac{\tilde{V}}{\tilde{V}_{,\phi}} F^2(\phi) d\phi \\ &= -\kappa \int_{\phi_b}^{\phi_e} \frac{1}{f(\phi)} \frac{d\phi}{\ln\left(\frac{U(\phi)}{f^2(\phi)}\right)'}\end{aligned}\quad (33)$$

If the original potential is quadratic $U(\phi) = \frac{1}{2}m^2\phi^2$, then (33) yields

$$\tilde{N} = -\frac{\kappa}{2} \int_{\phi_b}^{\phi_e} \frac{\phi d\phi}{1 + \sigma\phi^2} = \frac{\kappa}{4\sigma} \ln \left| \frac{1 + \sigma\phi_b^2}{1 + \sigma\phi_e^2} \right| \quad (34)$$

Neglecting ϕ_e as compared to $\phi_b \equiv \phi_{\text{max}}$, and using (11),

the number of e-folds in the Einstein frame is

$$\begin{aligned}\tilde{N} &\approx \frac{\kappa}{4\sigma} \ln |1 + \sigma \phi_{\max}^2| \\ &= \frac{\kappa}{4\sigma} \begin{cases} \ln |1 + \sin^2(\sqrt{\sigma} \tilde{\phi}_{\max})|, & \sigma > 0 \\ \ln |1 - \sinh^2(\sqrt{|\sigma|} \tilde{\phi}_{\max})|, & \sigma < 0 \end{cases} \quad (35)\end{aligned}$$

Note that in the limit $\sigma \rightarrow 0$, the number of e-folds reduces to that of the minimally coupled scalar field. Furthermore, considering the Taylor expansion of (35)

$$\frac{1}{\sigma} \ln |1 + \sigma x^2| = x^2 - \frac{1}{2} \sigma x^4 + \dots,$$

and the inequality $|\sinh(x)| \geq |\sin(x)|$, we conclude that, provided the same initial field, negative σ is preferable for successful inflation.

Following a similar derivation for a quartic original potential $U(\phi) = \frac{\lambda}{4} \phi^4$, we obtain the number of e-folds

$$\tilde{N} = \frac{\kappa}{8|\sigma|} \begin{cases} \sin^2(\sqrt{\sigma} \tilde{\phi}_{\max}), & \sigma > 0 \\ \sinh^2(\sqrt{|\sigma|} \tilde{\phi}_{\max}), & \sigma < 0 \end{cases} \quad (36)$$

As before, in the limit $\sigma \rightarrow 0$ the expression agrees with that of the minimally coupled model. Furthermore, a negative coupling constant again provides a greater number of e-folds.

As was mentioned before, the question as to how the inflaton gets at ϕ_{\max} is not addressed in the classical theory. In the next section, we shall estimate ϕ_{\max} on the grounds of LQC.

IV. ESTIMATION OF MAXIMUM INFLATON VALUE

In the framework of LQC, the self interacting scalar field, inflaton, arises as a microscopic vacuum fluctuation that is driven up its potential well by quantum effects. In the isotropic and homogeneous context, it is indeed quantum corrected (effective) Friedmann equations [7, 8] that have been shown to govern the evolution of the Universe in the semiclassical regime, when the scale factor $a_0 < a < a_*$. The boundedness of the geometrical density operator [15] manifests itself in the form of “repulsive” effects of gravity at Planckian scales [6] and leads to an anti-friction term in the effective Klein-Gordon equation [9].

Practically, the process of the inflaton’s ‘climbing-up’ can be broken into two stages: i) effective, driven by quantum modifications and consecutive ii) classical phase, conditioned by the field momentum gained in the effective regime. Both stages will be assumed to be *kinetic-dominated*, that is they occur relatively rapidly, and the potential terms, whenever they appear along with kinetic ones, are dominated by the latter. In other words, ϕ almost reaches its maximum before the potential effects become appreciable.

A. Effective phase

This phase starts with some initial conditions and ends when the scale factor $a \approx a_*$, a characteristic scale introduced in Eq.(39) below. We should say that the idea behind the estimate for \tilde{N} used in this paper is analogous to that in [16] except that the semiclassical phase ends *before* the inflaton reaches its maximum. Nor do we assume any specific asymptotic limit of the geometrical density spectrum.

The recipe provided by LQC as to how to proceed directly to an effective description is the following. In the Hamiltonian constraint, one is to replace all negative powers of triad p with appropriate factors of the spectrum of the inverse volume operator [17]

$$\frac{1}{p^{3/2}} \rightarrow d_j = \frac{D(q)}{p^{3/2}}, \quad (37)$$

where

$$\begin{aligned}D(q) &= q^{3/2} \left\{ \frac{8}{11} \left[(q+1)^{11/4} - |q-1|^{11/4} \right] \right. \\ &\quad \left. - \frac{8}{7} q \left[(q+1)^{7/4} - \text{sgn}(q-1) |q-1|^{7/4} \right] \right\}^6 \quad (38)\end{aligned}$$

and

$$q \equiv \frac{p}{p_*} \quad \text{with} \quad p_* \equiv a_*^2 := \frac{8\pi\gamma j \mu_0}{3} \ell_P^2 \quad (39)$$

The Planck length is defined as $\ell_P \equiv \sqrt{\kappa/8\pi} \equiv M_P^{-1}$. The asymptotic behavior of the density operator follows from the equation above. Not only does d_j remain bounded for $p \rightarrow 0$, it becomes proportional to a positive power of the scale factor. Specifically, $D(p/p_*) \propto p^{15/2}$ and $d \propto p^6$, thus making the matter Hamiltonian (38) well behaved. On the other hand, as p goes to infinity, one recovers the classical dependence: $D(p/p_*) \approx 1$ whereas $d \approx p^{-3/2}$.

Note, however, that this Hamiltonian constraint does not account for higher order perturbative corrections that proved to be important for small values of the parameter j [8, 18, 20]. More precisely, one has to consider the modified Hamiltonian

$$\mathcal{H}_{\text{eff}} = -\frac{3}{\kappa\gamma^2\mu_0^2} \sqrt{p} \sin^2(\mu_0 c) + \frac{1}{2} d_j(a) \pi^2 + V(\phi) \quad (40)$$

When $\mu_0 c \ll \pi/2$ one recovers the Hamiltonian (14). The discreteness corrections become important, when the matter part of the Hamiltonian is of order of the critical density

$$\rho_{\text{cr}} = 3/\kappa\mu_0^2\gamma^2 a^2 \quad (41)$$

corresponding to the maximum value of the gravitational part, attained when $\sin(\mu_0 c) = 1$.

Let us focus on the case, when the initial conditions are such that $\mu_0 c \ll \pi/2$. We shall investigate under what

restrictions this inequality will hold during the evolution within the effective regime. The effective counterparts of equations of motion (20) take the form

$$\begin{aligned}\dot{p} &= \sqrt{\frac{2\kappa}{3}} \sqrt{D(q) \frac{\pi^2}{p} + 2p^2 V(\phi)} \\ \dot{\phi} &= D(q) \frac{\pi}{p^{3/2}} \\ \dot{\pi} &= -p^{3/2} V'(\phi)\end{aligned}\quad (42)$$

As usual we have eliminated the connection variable using the constraint equation

$$\frac{c}{\gamma} = \sqrt{\frac{\kappa}{6}} \sqrt{\frac{\pi^2}{p^2} D(q) + 2pV(\phi)} \quad (43)$$

Note that within the effective regime, the field is very small ($\sigma\phi^2 \ll 1$) and the non-triviality of the coupling can be neglected. Let us therefore set $f(\phi)$ to unity.

From now on we take the quadratic potential

$$V(\phi) = \frac{1}{2} m^2 \phi^2, \quad (44)$$

m being the field mass. Consider the initial stage of the effective evolution, when the scale factor ranges from $p \approx p_0 \equiv \frac{4\pi\gamma\mu_0}{3}$ to $p = p_*$. For a small value of the inflaton (ϕ being of order 0.1), we see that in (43) the kinetic term overwhelmingly dominates over the potential term, provided the field mass $m = 10^{-6} M_P$. Thus for the sake of estimate of the maximum of $\mu_0 c$ we can discard the last term. Moreover, we notice that multipliers in the first term can be split into “fast” and “slow” ones. Formally, π is a slow variable, as the righthand side of $\dot{\pi}$ -equation in (42) is small due to a factor of m^2 in the potential term, whereas p is a fast variable. As far as $D(q)$ is a large positive power of p , the $D(q)$ -factor in (43) is the fastest. More rigorously, this statement can be cast as

$$\left| \frac{\dot{D}(q)}{D(q)} \right| \gg \left| \frac{\dot{p}}{p} \right| \gg \left| \frac{\dot{\pi}}{\pi} \right|. \quad (45)$$

Although, ϕ may become as fast as p , especially in the end of the effective phase, it is not used in the following derivation. The time derivative of c can be approximately written as

$$\frac{\dot{c}}{\gamma} \approx \sqrt{\frac{\kappa}{6}} \frac{d}{dt} \left(\frac{\pi}{p} \sqrt{D(q)} \right) \approx \sqrt{\frac{\kappa}{6}} \frac{\pi}{p} \frac{d(\sqrt{D(q)})}{dt} \quad (46)$$

This expression vanishes when

$$\dot{D} = \frac{dD(q)}{dp} \dot{p} = 0,$$

which is equivalent to

$$\frac{dD(q)}{dq} = 0$$

The solution to the last equation is $q_{\max} \approx 0.97$, which yields $\sqrt{D(q_{\max})}/q_{\max} \approx 1.2$. Neglecting the change in π and taking its initial value, we obtain the estimate for the maximum value of $\mu_0 c$

$$\mu_0 c_{\max} = \sqrt{\frac{\kappa}{6}} \frac{\sqrt{D(q_{\max})}}{q_{\max}} \mu_0 \gamma \frac{\pi_i}{p_*} = \frac{0.3}{j} \times \pi_i \quad (47)$$

This remarkable result tells us that, provided the aforementioned restrictions, the maximum value of $\mu_0 c$, and hence the maximum matter density, depends only on the initial field momentum, regardless the initial scale factor and field itself. Specifically ρ_{\max} is linearly proportional to π_i^2 . Thus the approximation $\mu_0 c \ll \pi/2$ remains valid throughout the effective regime, if

$$\pi_i \ll 5j\ell_P \quad (48)$$

Note that a similar derivation goes through, if the modified Hamiltonian (40) is used. At a given scale factor, the matter density must not exceed the critical value (41). The condition on the initial momentum (48) then becomes

$$\pi_i \leq \sqrt{\frac{2\kappa}{3D(q_{\max})}} \approx 3.45j\ell_P$$

This condition is not very restrictive and compatible with characteristic values of initial field momentum fluctuations derived from the Heisenberg uncertainty principle, provided initial field fluctuations of order $10^{-1} M_P$. For a more detailed analysis see [19].

Now suppose that the effective evolution starts at some initial data $p = p_i, \phi = \phi_i, \pi = \pi_i$. If π_i satisfies the condition (48), we can safely use the approximate Hamiltonian (37) and Hamilton equations (42). In order to estimate the maximum value of the inflaton at the end of the effective phase, it is convenient to rewrite the equations above in terms of the following dimensionless variables

$$(mt) \rightarrow t, \quad x := \frac{p}{p_*}, \quad y := \frac{\phi}{M_P}, \quad z := \frac{\pi}{mM_P p_*^{3/2}} \quad (49)$$

In these variables the equations of motion (42) take the form

$$\dot{x} = \frac{16\pi}{3} \sqrt{D(x) \frac{z^2}{x} + x^2 y^2} \quad (50)$$

$$\dot{y} = D(x) \frac{z}{x^{3/2}} \quad (51)$$

$$\dot{z} = -x^{3/2} y \quad (52)$$

We are going to get the approximate values for y and z (hence ϕ and π) at the end of the effective phase. In the same fashion, as was done before, we can integrate Eq. (51), treating z as a slow variable and fixing it at a constant value, that corresponds to z_i . Dividing (51) by (50), we express the derivative

$$\frac{dy}{dx} \approx \frac{D(x)}{x} \quad (53)$$

then the integration yields

$$y(x) \approx y_i + \sqrt{\frac{3}{2\kappa}} \int_{x_i}^x \frac{\sqrt{D(x')}}{x'} dx'. \quad (54)$$

Similarly

$$\frac{d(z^2)}{dx} \approx -\sqrt{\frac{3}{4\pi}} \frac{x^2 y(x)}{\sqrt{D(x)}} \quad (55)$$

Where we again have neglected the potential term in (50). Integrating we get

$$z^2(x) \approx z_i^2 - \sqrt{\frac{3}{4\pi}} \int_{x_i}^x \frac{x'^2 dx'}{\sqrt{D(x')}} \left(y_i + \sqrt{\frac{3}{16\pi}} \int_{x_i}^{x'} \frac{\sqrt{D(x'')}}{x''} dx'' \right) \quad (56)$$

Direct calculation shows that given reasonable initial conditions, the changes in the field and its momentum are indeed small and can be neglected when computing the value of the maximum matter density. In the dimensionful variables, the value of the inflaton at the end of the effective regime, ϕ_0 , is

$$\phi_0 \approx \phi_i + \sqrt{\frac{3}{2\kappa}} \int_{q_i}^{q_0} \frac{\sqrt{D(q)}}{q} dq \quad (57)$$

Assuming $q_i = \frac{1}{2j} = 0.1$ and $q_0 \approx 1$, the change in the scalar field is

$$\Delta\phi_{\text{eff}} \equiv \phi_0 - \phi_i = 0.14 M_{\text{P}} \quad (58)$$

The subscript ‘eff’ stands for effective. Remarkably the growth of the inflaton does not depend on its initial momentum at all!

We should now take the obtained values for p_0 and ϕ_0 , as well as $\pi_0 \approx \pi_i$, to be the initial data for the classical phase and estimate how high the inflaton will get.

B. Classical phase

When the scale factor significantly exceeds a_* , the quantum corrections become negligible and the evolution is very well described by the classical equations (20). Note, however, that we can no longer consider the inflaton to be small and should use the effective potential.

The end of the ‘climbing’ phase, i.e. the beginning of inflation, occurs at $\dot{\phi} \propto \pi = 0$, which does not correspond to any a-priori fixed scale factor. This is exactly the reason why one cannot use the technique of the previous sub-section. Instead one can again assume kinetic domination almost up to the top [21], where $\phi = \phi_{\text{max}}$. In other words, the inflaton changes insignificantly after the kinetic terms become less than the potential ones near ϕ_{max} . Note that we neglect the potential only when it appears along with a kinetic term; we should still keep

the right-hand side of the $\dot{\pi}$ -equation. In the light of this, we eliminate the time derivatives in Eqs. (20):

$$\begin{aligned} \frac{dp}{d\phi} &= \sqrt{\frac{2\kappa}{3}} p \\ \frac{d\pi}{d\phi} &= -\frac{p^3}{\pi} V'(\phi) \end{aligned} \quad (59)$$

Solving the first equation for the scale factor as a function of ϕ

$$p(\phi) = p_0 \exp \left\{ \sqrt{\frac{2\kappa}{3}} (\phi - \phi_0) \right\} \quad (60)$$

and substituting this expression into the second equation, we obtain

$$\frac{d\pi}{d\phi} = -\frac{p_0^3}{\pi} \exp \left\{ \sqrt{6\kappa} (\phi - \phi_0) \right\} V'(\phi) \quad (61)$$

Finally, separating the variables in the last equation and integrating it from $\pi = \pi_0$ (beginning of the classical climbing) to $\pi = 0$ (end of the classical climbing), we arrive at a transcendental equation for ϕ_{max} :

$$\frac{\pi_0^2}{2p_0^3} = \int_{\phi_0}^{\phi_{\text{max}}} \exp \left\{ \sqrt{6\kappa} (\phi - \phi_0) \right\} V'(\phi) d\phi \quad (62)$$

Before giving numerical solutions, let us draw some analytical conclusions from this equation. As the effective potential is steeper for a positive σ , we should expect a smaller maximum value of the inflaton in this case than for a negative coupling. Note also that in the integrand of (62) the exponential is the fastest growing function, in the same sense as before. Therefore, when integrating, the potential factor can be taken out of the integral and evaluated, for the sake of estimate, at the upper limit. We now restrict our attention to the case of quadratic original potential and consider the Taylor expansion of the effective potential

$$V(\phi) = \frac{1}{2} m^2 \phi^2 \left(1 + \frac{5}{3} \sigma \phi^2 + O(\phi^4) \right) \quad (63)$$

Then the transcendental equation (62) can be approximately rewritten as

$$\frac{9}{5y^3} (Ae^{-y} - y) = \frac{\sigma}{\kappa} \quad (64)$$

where we have defined

$$y := \sqrt{6\kappa} (\phi_{\text{max}} - \phi_0), \quad A := \frac{3\kappa\pi_0^2}{m^2 p_0^3} \quad (65)$$

Let us first set σ to zero and denote the corresponding solution \bar{y} , i.e. \bar{y} satisfies

$$\bar{y} e^{\bar{y}} = A \quad (66)$$

The solution to this equation is the Lambert function $\bar{y} = W(A)$. It is plotted in Fig. 2 as a function of the

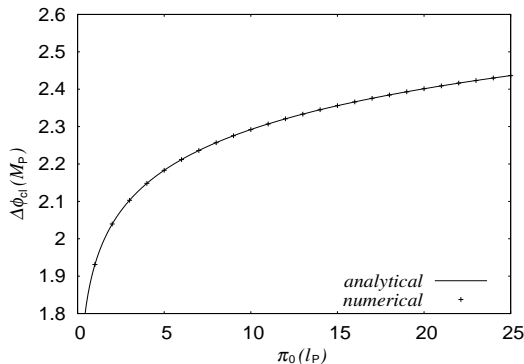


FIG. 2: Maximum change of the inflaton as a function of the initial field momentum for $\sigma = 0$ and $p_0 = p_*$. The analytical curve is the Lambert function defined in (66), whereas the numerical points are obtained as solutions to the equations of motion (42).

initial field momentum for a fixed initial scale factor $p_0 = p_*$. We see that dependence of $\phi_{\max} - \phi_0$ upon π_0 is close to logarithmic, i.e it is not very sensitive to initial momentum. For instance, if $\pi_0 = 5\ell_P$, the inflaton grows by about two Planck masses during the effective phase. This is insufficient for successful inflation.

Let us now take a non-zero coupling constant and, treating it as a small parameter, consider the corrections to the maximum value of the scalar field $\epsilon := y - \bar{y}$. The equation (64) then yields

$$\epsilon = -\frac{5\sigma}{9\kappa}\bar{y}^2 = -\frac{10}{3}\sigma(\bar{\phi}_{\max} - \phi_0)^2 \quad (67)$$

The first important consequence of this expression is that the σ -corrections to ϕ_{\max} are small and proportional to *minus* σ . In other words, in the case of a negative coupling strength, the inflaton indeed gets higher up the potential hill.

Before combining all the contributions to ϕ_{\max} , we note that there also takes place an effective (super-) inflation, while the universe is expanding from $a = a_i$ to $a = a_*$ [7]. This also provides a couple of e-folds, as the scale factor $a \equiv \sqrt{p}$ increases by a factor of $\sqrt{\frac{p_*}{p_i}} \equiv \sqrt{2j}$.

According to (35), $\phi_{\max} \approx 2.6M_P$ is sufficient to provide 60 e-folds. Let us now give a numerical estimate for the maximum inflaton value. For $j = 5$, taking the initial conditions $p_i = p_*/2j = 0.1p_*$, $\phi_i = 0.1M_P$, $\pi_i = 5\ell_P$, the effective growth of the field is $0.14M_P$, whereas the classical contribution is $2.25M_P$. Adding everything up we get $\phi_{\max} \approx 2.5M_P$. We shall discuss how generic this result is and how it varies with a different choice of initial conditions in the next section.

V. NUMERICAL RESULTS AND DISCUSSION

In this section, we study numerical solutions to the differential equations (42). Again we will be mostly in-

terested in the maximum value of the inflaton for the allowed range of initial conditions and the parameter σ .

Let us first discuss the initial field and the scale factor. As was shown in the previous section, the dependence on the former is rather trivial: the initial value ϕ_0 acts as constant of integration and should be merely added to the change in the inflaton during the climbing phase.

The dependence of ϕ_{\max} on the initial scale factor is more complicated, but the value of p_i itself is quite restricted. It appears that there is a natural choice of p_i , associated with the smallest eigenvalue of the area operator [22], related to μ_0 . In fact, as one can see from (57), the effective growth of the scalar $\Delta\phi_{\text{eff}}$ depends on the limits of integration, hence scale factor, only through the ratio $q \equiv \frac{p}{p_*}$. The lower limit is given by $q_i = \frac{1}{2j}$, while the upper limit is not fixed a-priori. It has a physical meaning of the marginal value that separates the effective and classical behavior of the spectrum of the geometrical density operator. For the estimate in the previous section we used $q = 1$ as the upper limit. A closer look at the graph of $D(q)$ (see Fig. 3) shows that there is a maximum around $q \approx 1$. In other words, the behavior of the spectrum is not yet classical. It would be more reasonable to stop the integration (which is meant to be over the effective domain) at a somewhat greater value of q , where $D(q)$ is essentially close to unity. Recall that $q_0 = 1$ was giving $\Delta\phi_{\text{eff}} = 0.14M_P$. At the same time, analysis of Eq. (54), for $\Delta\phi_{\text{eff}}$ as a function of the upper limit of integration q_0 , shows that the effective growth of the inflaton sensitively depends on q_0 , whereas the classical formula (66) indicates a much weaker dependence upon q_0 . For instance, taking $q_0 = 2$, one would get $\Delta\phi_{\text{eff}} = 0.3M_P$, while $q_0 = 4$ yields $\Delta\phi_{\text{eff}} = 0.5M_P$. This implies that if one increases q_0 , one should expect a somewhat greater value of total $\Delta\phi \equiv \Delta\phi_{\text{eff}} + \Delta\phi_{\text{cl}}$. Nevertheless, for the sake of estimate, we will stick with $q_0 = 1$ and compare the numerical results with analytical formulas.

Summing up, as the effective growth of the inflaton is not very substantial and does not depend on the initial field momentum π_0 or coupling strength σ , the quantity of the most interest is the classical part of $\Delta\phi$. The latter is determined by π_0 and σ .

The dependence of $\Delta\phi_{\text{cl}}$ on the initial momentum p_0 for five different values if σ is displayed in Fig. 4. The curves correspond (top to bottom) to $\sigma/\ell_P^2 = -0.1, -0.05, 0, 0.05$ and 0.1 and are plotted for the allowed range of initial momenta: $\frac{\pi_0}{\ell_P} < 5j = 25$. The maximum inflaton value grows with initial momentum and qualitatively resembles the Lambert function. It is steepest for small momenta and flattens out when π_0 becomes large. Quantitatively, for $\sigma = 0$ there is a very good agreement with the analytical result (66) and the graph of Fig. 2. The curves, corresponding to opposite values of the coupling strength, are situated symmetrically around the minimal curve, which justifies the linear (in σ) approximation used in (63). Furthermore, direct calculation shows that the σ -corrections to the central

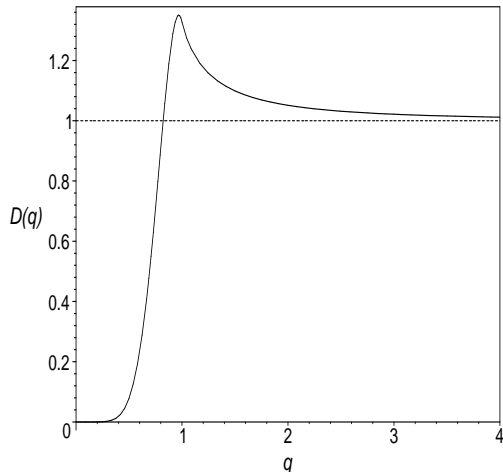


FIG. 3: The spectrum of the geometrical density operator $D(q)$ introduced in Eq. (38)

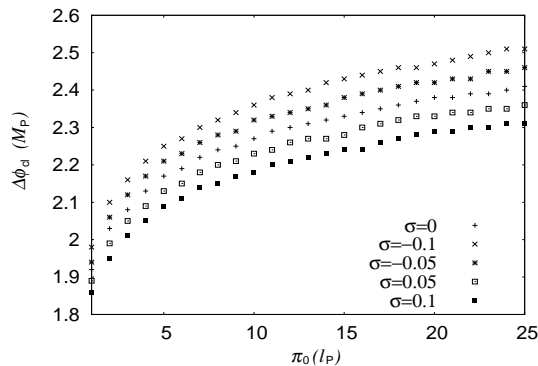


FIG. 4: The change of the inflaton during the classical phase as function of its initial momentum for different values of σ (measured in the units of ℓ_P^2).

curve are very close to those predicted by formula (67) for both small and large initial momenta.

Classical climbing brings the scalar field fairly close to the required values determined by $\tilde{N} = 60$ and (35). For instance, $\phi_{\text{req}} = 2.6M_P$ for $\sigma = -0.05\ell_P^2$ and $\phi_{\text{req}} = 2.5M_P$ for $\sigma = -0.10\ell_P^2$. However, $\Delta\phi_{\text{cl}}$ on its own is not enough for sufficient inflation. At the same time, the number of e-folds is very sensitive to deviations in ϕ_{max} near the required values. In other words, even small (positive) variations of ϕ_i and/or $\Delta\phi_{\text{eff}}$, such as of order of $0.1M_P$, would lead to sufficient \tilde{N} and should be seriously taken into account.

We see from Fig. 4 that a negative coupling constant indeed gives a greater change of the inflaton, and more negative values are preferable for successful inflation. This fact can be also understood from the Jor-

danian perspective. The non-minimally coupled action (1) can be thought of as a standard one with a field-dependent gravitational constant (2). Clearly, a negative σ results in ‘weaker’ gravitational coupling, and it is not surprising that the inflaton would reach to a greater value (yielding a greater N), than for $\sigma = 0$. Moreover, if $\sigma < 0$, the second term of the righthand side of the relation between the number of e-folds in the Einstein and Jordan frames (19) is of order one and can be neglected. Thus $\tilde{N} \approx N$ and is greater than in the case of minimal coupling. Similar considerations work for a positive coupling as well and imply a smaller number of e-folds.

We should now clarify the seeming discrepancy with the result of Futamase and Maeda [4]. As we have already mentioned, in that paper, the authors argued that, in order to allow successful inflation, the coupling constant ξ had to be fine-tuned within a very narrow interval: $\xi < 10^{-12}$ for $\xi > 0$ and $|\xi| < 10^{-3}$ for $\xi < 0$. These restrictions on ξ arose from the heuristic argument based on Linde’s chaotic inflationary scenario. The values of ϕ_i are assumed to be randomly distributed from zero up to Planckian energy densities $V(\phi) \sim M_P^4$. Such potential is attained at $\phi \sim 10^6 M_P$. The latter must necessarily be less than the critical value ϕ_{crit} (4), which implies $\sigma \sim 10^{-12}\ell_P^2$ and $\xi \sim 10^{-14}$.

The above conditions can be relaxed in the framework of LQC. As the initial values of the scalar field appear as vacuum fluctuations, that are amplified during the ‘climbing’ phase, one just needs to restrict the coupling constant so that ϕ_{crit} merely exceeds $\phi \sim 3M_P$ - the maximum inflaton value we are interested in. That yields $\sigma \sim 10^{-1}\ell_P^2$ and $\xi \sim 10^{-3}$. The former is exactly the maximum value of the parameter σ we have considered in the paper.

To summarize, the main characteristic of inflation, the number of e-folds, depends on the initial matter fluctuations and the coupling parameter σ . The coupling strength is bounded to be less than $0.1\ell_P^2$ by the condition $\phi_{\text{max}} < \phi_{\text{crit}} \equiv 1/\sqrt{|\sigma|}$ for both positive and negative couplings. At the same time, the most negative σ would work best for successful inflation. The restriction on the initial field momentum π_i appears as a requirement for the matter density to remain subcritical and implies $|\pi_i| < 5j$. With these restrictions, the value of the inflaton will increase by approximately $2.0 - 2.6M_P$ during the ‘climbing’ (effective and classical) phase. Together with the initial vacuum fluctuations of the scalar field of order of several tenths of M_P this would lead to $\tilde{N} \geq 60$, i.e. sufficient inflation. It should be noted that one cannot get \tilde{N} much greater than 60, which indicates that observations may be sensitive to the mechanism discussed here.

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